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Some New Exact Solutions of The Space Time Boussinesq and Kdv Fractional Partial Differential Equation

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Article Details

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ABSTRACT

In this paper, the Modified Riemann-Liouville derivative is proposed to solve Space Time Boussinesq Fractional Partial Differential Equation, and Jumarie's modified Riemann Liouville derivative is used to convert nonlinear partial fractional differential equation to nonlinear ordinary differential equations. The modified Kudryashov method is applied to compute an approximation to the solutions Of the Space Time Boussinesq Fractional Partial Differential Equation and some Solutions of Space Time Korteweg–de Vries Fractional Partial Differential Equations. As a result, many exact solutions of fractional partial differential equations arising in plasma physics in the sense of modified Riemann–Liouville derivative.

INTRODUCTION

The fractional partial differential equation seems more and more often in different fractional Navier-Stokes equation, like Ginzberg- Landau equation, Burgers equation, Langevin equation and Schrodinger equation [1-3].The study of exact solution in the field of partial differential equations has great importance in different scientific and engineering disciplines. Among the various class of PDEs, the classical Boussinesq [4] is a prominent model that has an ability to easily captures the dynamics of complex systems with memory effects and non-local interactions. In today's world, researchers have brought revolution in the field of Mathematics by using this approach, contributing to both theoretical advancements and practical applications. The Boussinesq FPDE is more advance form of the classical Boussinesq equation which includes fractional derivatives to define non-local connections and memory effects within the system under consideration. This equation is more suitable for all those problems where traditional PDE models fail to give solutions. Its applications are in the field of fluid dynamics, nonlinear waves, heat transfer, and various other fields where traditional PDE models fall short in capturing intricate phenomena [4].

Korteweg de Vries (KdV) equation was first introduced by Korteweg de Vries [5]. KdV equation has great impact in various nonlinear phenomena in physics and applied mathematics such as solid-state physics, plasma physics, particle acoustic wave, fluid mechanics, quantum field and so on [6,7]. Nonlinear fractional partial differential equation (NLFPDEs) is applicable for mathematical modeling of many complex and physical phenomena.

Among these, the Space-Time Boussinesq and Korteweg–de Vries (KdV) equations hold particular interest due to their role in describing phenomena such as shallow water waves and plasma waves. However, solving these FPDEs analytically remains challenging due to their nonlinearity and the non-local properties introduced by fractional calculus.

In particular, Jumarie's modified Riemann-Liouville derivative has proven effective in transforming nonlinear FPDEs into more manageable nonlinear ordinary differential equations (ODEs), facilitating the search for exact solutions. This transformation is critical as it allows researchers to apply various analytic methods designed for ODEs, thereby expanding the available solution techniques for FPDEs.

In this paper, we employ the modified Kudryashov method to approximate solutions for the

Space-Time Boussinesq Fractional Partial Differential Equation, aiming to identify new exact solutions for equations central to plasma physics. The modified Kudryashov method has demonstrated its effectiveness in handling nonlinear FPDEs, providing a structured approach to finding solutions with a high degree of accuracy. By applying this method, we obtain a suite of exact solutions for both the Space-Time Boussinesq and KdV FPDEs, showcasing the versatility and power of the modified Riemann-Liouville derivative in the context of fractional calculus.

Our findings contribute to the growing body of research on exact solutions of fractional models, potentially enhancing the understanding of wave phenomena in plasmas and other complex systems where traditional models fall short.

PRELIMINARIES AND THE MODIFIED KUDRYASHOV METHOD & MODIFIED RIEMANN-LIOUVILLE DERIVATIVE

The Jumaries modified Riemann-Liouville derivative of order α is defined by the expression

$$D_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \int_0^x (x-\xi)^{-\alpha-1}[f(\xi)-f(0)]d\xi, \qquad 2.1$$

if $\alpha < 0$,

$$D_x^\alpha f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-\xi)^{-\alpha}[f(\xi)-f(0)]d\xi \qquad 2.2$$

if $0 < \alpha < 1$,

$$D_x^\alpha f(x) = \left(f^{(n)}(x)\right)^{(\alpha-n)} \qquad 2.3$$

if $n \leqslant \alpha \leqslant n+1, n \geqslant 1$,

where $f: R \to R$ is a continuous function

Some properties of the fractional modified Riemann-Liouville derivative were summarized in, three useful formulas of them are

$$D_x^\alpha x^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} x^{\gamma-\alpha} \qquad 2.4$$

$$\gamma > 0$$

$$D_x^\alpha (u(x)(x)) = (x) D_x^\alpha u(x) + u(x) D_x^\alpha (x) \qquad 2.5$$

$$D_x^\alpha [f(u(x))] = f'_u(u) D_x^\alpha u(x) = D_x^\alpha f(u) \left(u'_x\right)^\alpha \qquad 2.6$$

which are direct consequences of the equality

$D_x^{\alpha} x(t) = \Gamma\,(1+\alpha) dx(t)$ 2.7

The main steps of the simplest equation method are summarized as follows:

METHODOLOGY

STEP 2.1:

Consider a general form of the time fractional partial differential equation

$$P\left(u, D_t^{\alpha} u, u_x, D_t^{3\alpha} u, u_{xx}, \ldots\right) = 0 \quad 2.8$$

STEP 2.2:

To find the exact solution of Eq. (2.8) we introduce the variable transformation [8]

$$\frac{dQ}{d\xi} = \mathrm{a}Q(\xi) + \mathrm{b}Q^2(\xi) \quad 2.9$$

$$u(x,t) = y(\xi), \xi = lx - \frac{\lambda}{\Gamma(1+\alpha)} t^{\alpha} \quad 2.10$$

where l and λ are constants to be determined later. Using Eq. (2.8) changes the Eq. (2.7) to an ODE

$$Q\left(y, \frac{\partial y}{\partial \xi}, \frac{\partial^2 y}{\partial \xi^2}, \ldots\right) = 0 \quad 2.11$$

where $y = y(\xi)$ is an unknown function, Q is a polynomial in the variable y and its derivatives.

EXAMPLE 3.1

The Boussinesq equation is a nonlinear partial differential equation (PDE) that defines the spread of long surfs in shallow sea, among other phenomena:[4]

Consider the 4th order Boussinesq Fractional Partial Differential Equation [4]:

$D_t^{\alpha\alpha} u(x,t) + \delta D_x^{\beta\beta} u^2(x,t) + v D_x^{\beta\beta} u(x,t) + d D_x^{\beta\beta\beta\beta} u(x,t) = 0$ 3.1

where α, β are order of derivative and $\alpha = \beta$ & $0 < \alpha < 1,\ 0 < \beta < 1$,

δ, v, d are constant

Using variable transformation in equation in (3.1) we get this

$$-\lambda u''(\xi) + 2\,\delta\, l^2 . u(\xi) . u''(\xi) + v l^2 u''(\xi) + l^2 d u'^{v}(\xi) = 0 \quad 3.2$$

By balancing the order between the highest order derivative term and nonlinear term in Eq. (3.2) we can obtain M = 2.

$$u(\xi) = a_0 + a_1 Q + a_2 Q^2 \quad 3.3$$

The above is the nonlinear equation (3.3&3.2). Solve this system for the following parameters

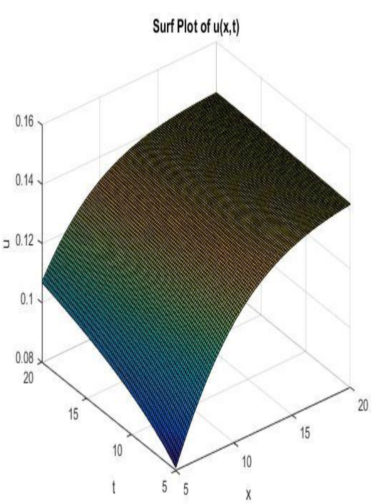
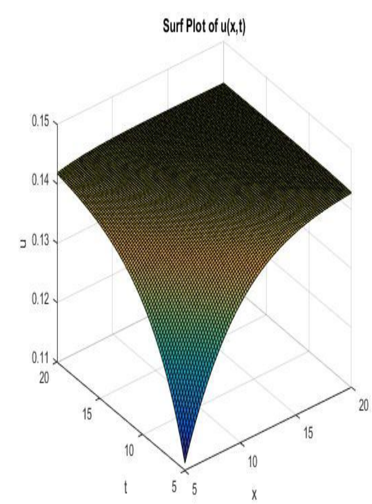
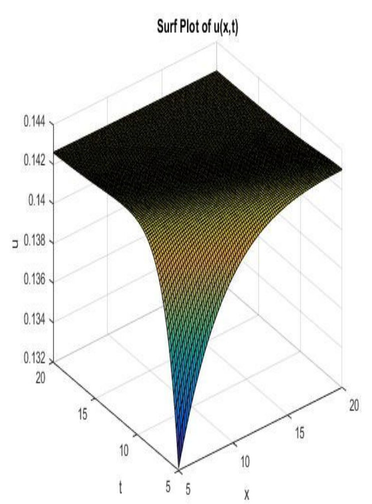
$a_o,\ a_1,\ a_2, and\ \lambda$.We get finally two cases.

CASE-1

$$a_0=\frac{(32dl^2(\ln a)^2)}{\delta},a_1=\frac{6dl^2(\ln a)^2)}{\delta},\ a_2=-\frac{6dl^2(\ln a)^2)}{\delta},\lambda=l\left(-u-65dl^2(\ln a)^2\right)^{\frac{1}{2}})$$

$$u(x,t)=\Bigg(2dl^2(\ln a)^2\left(35a^{lx}a^{\left(lt^\alpha(-v-65l^2(\ln a)^2)^{\frac{1}{2}}\right)}\right)/\left(\Gamma(1+\alpha)\right)d_1 + \left(16a^{2lx}a^{(2lt^2}(-4-65dl^2(\ln a)^2)^{\frac{1}{2}}\right)/\left(\Gamma(1+\alpha)\right)d_1^2 + 16\ \Bigg)\left(\delta\, a^{lx}\left(\frac{lt^\alpha(-65dl^2(\ln a)^2-v)^{\frac{1}{2}}}{\left(\Gamma(1+\alpha)\right)d_1}+l\right)\right)^2 \quad 3.4$$

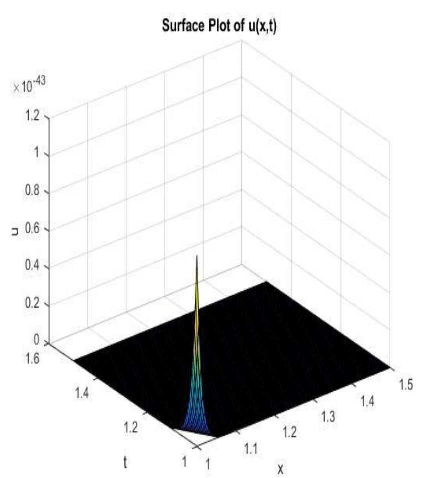
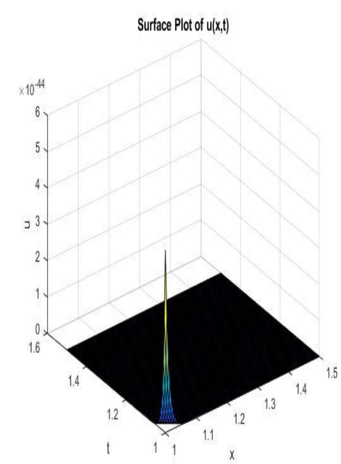
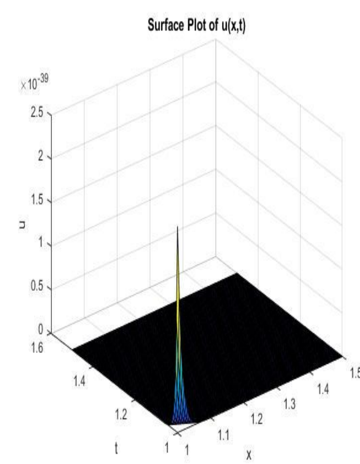
The distribution function u (x, t) in eq (3.4) as a 3-dimensions graph for α is different but parameter is same d=0.9, l=0.2, a=0.56, d1=2.0, v=-10, δ=1

α =0.3	**α =0.6**	**α =0.9**
		

Case–2 $a_0=\frac{(32dl^2(\ln a)^2)}{\delta},a_1=\frac{6dl^2(\ln a)^2)}{\delta},\ a_2=-\frac{6dl^2(\ln a)^2)}{\delta},\lambda=-\ l\left(-u-65dl^2(\ln a)^2\right)^{\frac{1}{2}})$

$$u(x,t)=\left(2dl^{2}(\ln a)^{2}\left(16a^{2lx}d_{1}^{2}+16a^{((\,2lt^{\alpha}\left(\left(-\mathrm{v}-65dl^{2}\ (\ln a)^{2}\right)^{\frac{1}{2}}\right)}/\left(\Gamma(1+\alpha)\right)\right)+\frac{35a^{lx}a^{\left(lt^{\alpha}\left(-\mathrm{v}-65dl^{2}\ (\ln a)^{2}\right)^{\frac{1}{2}}\right)/\left(\Gamma(1+\alpha)\right)}d_{1})}{\left(\frac{\delta\left(a^{lt^{\alpha}}\left(-65dl^{2}(\ln a)^{2}-u\right)^{\frac{1}{2}}\right)}{\left(\Gamma(1+\alpha)\right)}+\alpha^{lx}d_{1}\right)^{2}}\right) \quad 3.5$$

The distribution function u (x, t) in eq (3.5) as a 3-dimensions graph for α is different but parameter is same d=0.9, l=0.5, a=2.56, d1=2.50, u=-15.5, δ=1

α =0.3	α =0.6	α =0.9
		

Example 3.2

Let us first consider the space-time modified KdV equation [9].

$$D_{t}^{\alpha}\varphi(x,t)+\mathrm{v}\varphi^{1/2}(x,t)D_{x}^{\beta}\varphi(x,t)+\delta D_{x}^{\beta\beta\beta}\varphi(x,t)=0 \quad 3.6$$

$$0<\alpha<1,\,0<\beta<1$$

V and δ is constant

Fractional KdV that define the nonlinear propagation of dust ion acoustic (DIA) solitary wave.

To start with, suppose $\alpha=\beta$

Using variable transformation in equation in (3.6) we get this

$$-2\lambda y(\xi)\cdot y'(\xi)+v\cdot y(\xi)\cdot\left(2ly(\xi)\cdot y'(\xi)\right)+\delta\left(2l^3\left[3y'(\xi).y''(\xi)+y(\xi)y'''(\xi)\right]\right)=0 \qquad 3.7$$

By balancing the order between the highest order derivative term and nonlinear term in Eq. (3.7) we can obtain M = 2

$$\mathrm{y}(\xi)=a_0+a_1\mathrm{z}+a_2z^2 \qquad 3.8$$

The above is the nonlinear equation (3.7&3.8). Solve this system for the following parameters $a_o,\ a_1,\ a_2,\ a, and\ b$.We get finally three cases.

Case-1 $\boldsymbol{a}=\dfrac{\sqrt{30}.v.\sqrt{(-\frac{\lambda}{l.v})}.\sqrt{(-\frac{15}{2.\delta.v})}}{30.l},\boldsymbol{b}=-\dfrac{v.\sqrt{\left(-\frac{15}{2.\delta.v}\right)}}{15.l},\boldsymbol{a_0}=\mathbf{0},\boldsymbol{a_1}=-\dfrac{\sqrt{30}.\sqrt{\left(-\frac{\lambda}{l.v}\right)}}{2},\boldsymbol{a_2}=\mathbf{1}$

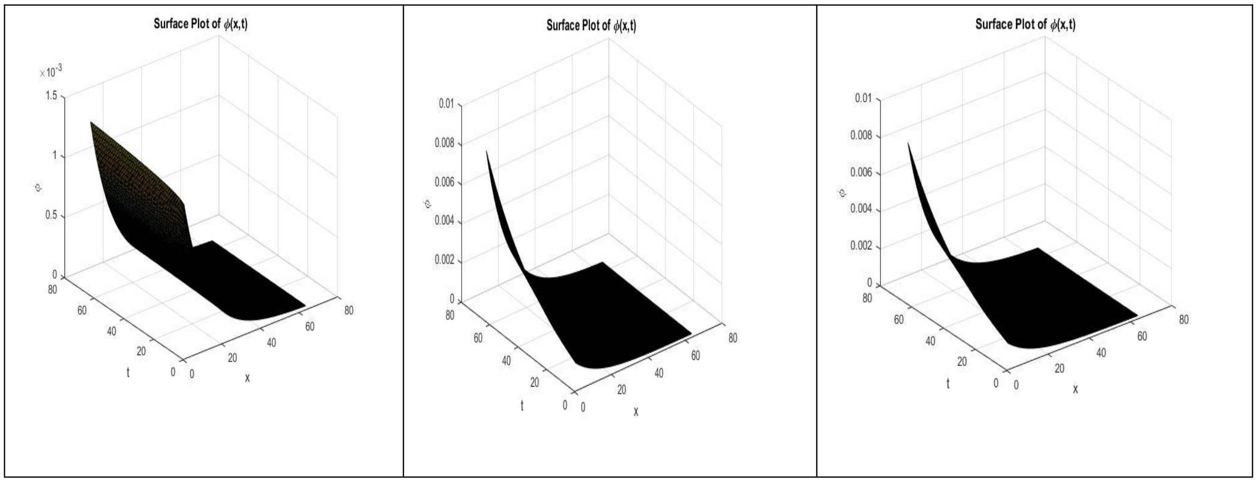
$$\varphi(x,t)=\frac{15.\delta.\boldsymbol{\lambda}v.e^{\left(\frac{\sqrt{30}.v.\sqrt{\left(-\frac{\lambda}{l.v}\right)}.\left(\omega_o+l.x-\frac{\lambda t^\alpha}{\Gamma(1+\alpha)}\right).\sqrt{\left(-\frac{15}{2.\delta.v}\right)}}{30.l}\right)}.\sqrt{-\frac{15}{2.\boldsymbol{\delta}.v}}}{(30\boldsymbol{\delta}l^2-v.e^{\left(\frac{\sqrt{30}.v.\sqrt{\left(-\frac{\lambda}{l.v}\right)}.\left(\omega_o+l.x-\frac{\lambda t^\alpha}{\Gamma(1+\alpha)}\right).\sqrt{\left(-\frac{15}{2.\delta.v}\right)}}{15.l}\right)}+} \qquad 3.9$$

$$\frac{.\sqrt{(-\frac{15}{2.\boldsymbol{\delta}.v})))}}{4\boldsymbol{\delta}.l.v.\mathrm{e}^{\left(\frac{\sqrt{30}.v.\sqrt{\left(-\frac{\lambda}{l.v}\right)}.\left(\omega_o+l.x-\frac{\lambda t^\alpha}{\Gamma(1+\alpha)}\right).\sqrt{\left(-\frac{15}{2.\delta.v.}\right)}.}{30.l}\right)}.\sqrt{-\frac{15}{2.\boldsymbol{\delta}.v})}})$$

The distribution function $\varphi(x,t)$ in eq 3.9 as a 3-diemensional graph for the distribution function α is different but parameter are same $l=39.5$, $a=32.56$, $\delta=19.9$,

$w_0=20.5$, $v=-8$, $\boldsymbol{\lambda}=21$,

$\boldsymbol{\alpha}$ =0.3	$\boldsymbol{\alpha}$ =0.6	$\boldsymbol{\alpha}$ =0.9



Case-2

$$a = -\frac{\sqrt{30}.v.\sqrt{(\frac{\boldsymbol{\lambda}}{l.v})}.\sqrt{(-\frac{15}{2.\delta.v})}}{30.l}, b = \frac{-v.\sqrt{(-\frac{15}{2.\delta.v})}}{15.l}, a_0 = \frac{5c}{4lv}, a_1 = \frac{\sqrt{30}.\sqrt{\frac{\boldsymbol{\lambda}}{l.v}}}{2}, a_2 = 1$$

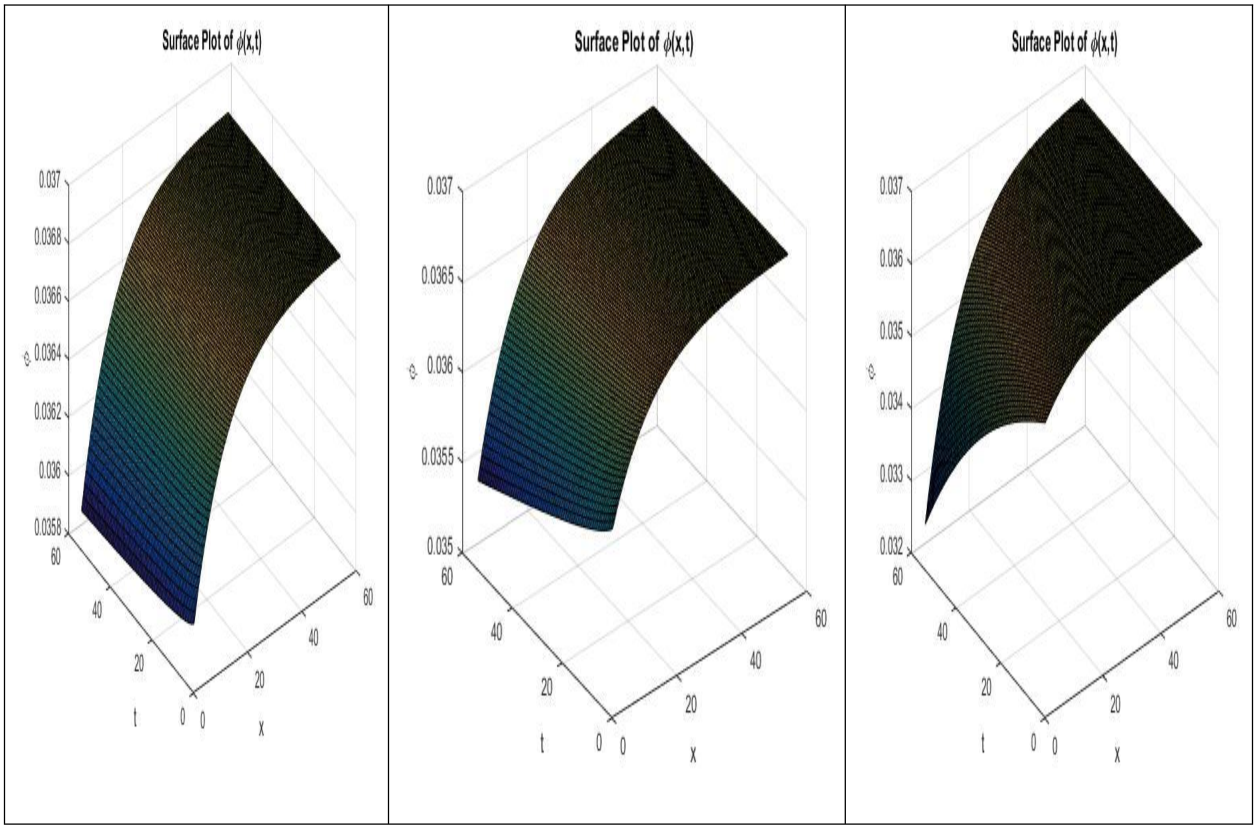
$$\varphi(x,t) = -\frac{5.\boldsymbol{\lambda}(v - 30\boldsymbol{\delta}.l^2.e^{\left(\frac{\sqrt{30}.v.\sqrt{\left(\frac{\boldsymbol{\lambda}}{l.v}\right)}.\left(\omega_o + l.x - \frac{\boldsymbol{\lambda}t^{\alpha}}{\Gamma(1+\alpha)}\right).\sqrt{\left(-\frac{15}{2.\delta.v}\right)}}{15.l}\right)} +}{4.l.v(30\boldsymbol{\delta}l^2.e^{\left(\frac{\sqrt{30}.v.\sqrt{\left(\frac{\boldsymbol{\lambda}}{l.v}\right)}.\left(\omega_o + l.x - \frac{\boldsymbol{\lambda}t^{\alpha}}{\Gamma(1+\alpha)}\right).\sqrt{\left(-\frac{15}{2.\delta.v}\right)}}{15.l}\right)} - v} \quad 3.10$$

$$\frac{8\boldsymbol{\delta}.lv.e^{\left(\frac{\sqrt{30}.v.\sqrt{\left(\frac{\boldsymbol{\lambda}}{l.v}\right)}.\left(\omega_o + l.x - \frac{\boldsymbol{\lambda}t^{\alpha}}{\Gamma(1+\alpha)}\right).\sqrt{\left(-\frac{15}{2.\delta.v.}\right)}.}{30.l}\right)}.\sqrt{-\frac{15}{2.\boldsymbol{\delta}.v}}}{4\boldsymbol{\delta}.l.v.e^{\left(\frac{\sqrt{30}.v.\sqrt{\left(\frac{\boldsymbol{\lambda}}{l.v}\right)}.\left(\omega_o + l.x - \frac{\boldsymbol{\lambda}t^{\alpha}}{\Gamma(1+\alpha)}\right).\sqrt{\left(-\frac{15}{2.\delta.v.}\right)}.}{30.l}\right)}.\sqrt{\left(-\frac{15}{2.\boldsymbol{\delta}.v}\right)}}$$

The distribution function $\varphi(x,t)$ in eq 3.10 as a 3-diemensional graph

for α is different but parameter are same $l = 39.5$, $\delta = 19.9$, $w_0 = 20.5$, $v = 18$, $\boldsymbol{\lambda} = 21$,

$\boldsymbol{\alpha}$ =0.3	$\boldsymbol{\alpha}$ =0.6	$\boldsymbol{\alpha}$ =0.9



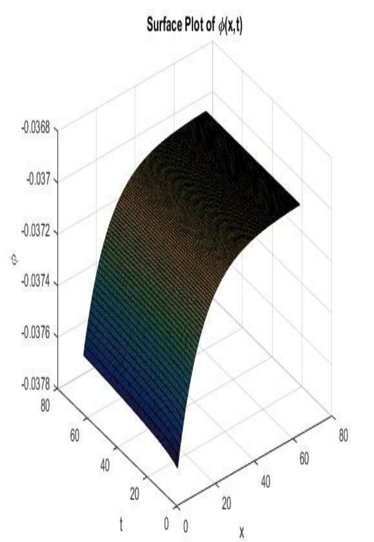
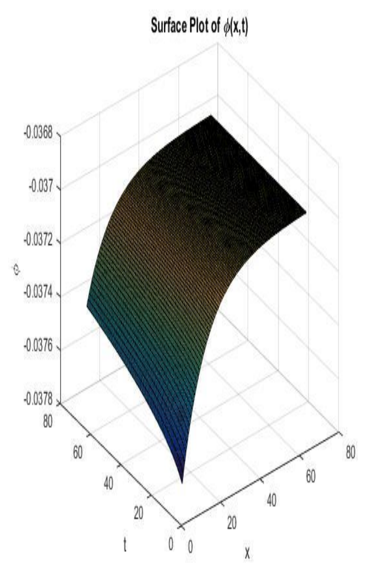
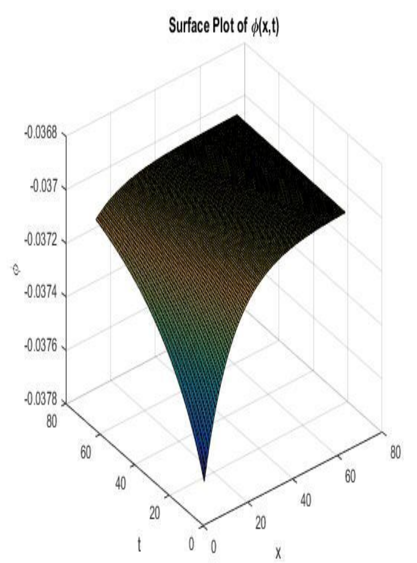
Case-3

$$a=\frac{\sqrt{30}.v.\sqrt{(-\frac{\boldsymbol{\lambda}}{l.v})}.\sqrt{(-\frac{15}{2.\delta.v})}}{30.l}, b=-\frac{v.\sqrt{(-\frac{15}{2.\delta.v})}}{15.l}, a_0=\frac{5\boldsymbol{\lambda}}{4lv}, a_1=\frac{\sqrt{30}.\sqrt{(-\frac{\boldsymbol{\lambda}}{l.v})}}{2}$$

$$\varphi(x,t)=-\frac{5.\boldsymbol{\lambda}(30\boldsymbol{\delta}.l^2.e^{\left(\frac{\sqrt{30}.v.\sqrt{\left(\frac{\boldsymbol{\lambda}}{l.v}\right)}.\left(\omega_o+l.x-\frac{\boldsymbol{\lambda}t^{\alpha}}{\Gamma(1+\alpha)}\right).\sqrt{\left(-\frac{15}{2.\delta.v}\right)}}{15.l}\right)}-v}{4.l.v(v-30\boldsymbol{\delta}l^2.e^{\left(\frac{\sqrt{30}.v.\sqrt{\left(\frac{\boldsymbol{\lambda}}{l.v}\right)}.\left(\omega_o+l.x-\frac{\boldsymbol{\lambda}t^{\alpha}}{\Gamma(1+\alpha)}\right).\sqrt{\left(-\frac{15}{2.\delta.v}\right)}}{15.l}\right)}}+ \qquad 3.11$$

$$\frac{8\boldsymbol{\delta}.lv.e^{\left(\frac{\sqrt{30}.v.\sqrt{\left(\frac{\boldsymbol{\lambda}}{l.v}\right)}.\left(\omega_o+l.x-\frac{\boldsymbol{\lambda}t^{\alpha}}{\Gamma(1+\alpha)}\right).\sqrt{\left(\frac{15}{2.\delta.v.}\right)}.}{30.l}\right)}.\sqrt{(-\frac{15}{2.\boldsymbol{\delta}.v})})}{4\boldsymbol{\delta}.l.v.e^{\left(\frac{\sqrt{30}.v.\sqrt{\left(\frac{\boldsymbol{\lambda}}{l.v}\right)}.\left(\omega_o+l.x-\frac{\boldsymbol{\lambda}t^{\alpha}}{\Gamma(1+\alpha)}\right).\sqrt{\left(-\frac{15}{2.\delta.v.}\right)}.}{30.l}\right)}.\sqrt{\left(-\frac{15}{2.\boldsymbol{\delta}.v}\right)})}$$

The distribution function $\varphi(x,t)$ in eq 3.11 as a 3-diemensional graph for α is different but parameter are same $l = 39.5$, $\delta = 19.9$, $w_0 = 20.5$, $v = 18$, $\boldsymbol{\lambda} = 21$,

$\boldsymbol{\alpha}$ =0.3	$\boldsymbol{\alpha}$ =0.6	$\boldsymbol{\alpha}$ =0.9
		

CONCLUSION

In this dissertation, the modified Kudryashov method is used to find exit solutions of the space time Boussinesq fractional partial differential equation and the Korteweg–de Vries (KdV) partial differential equation. by using RL approaches. Numerical simulation has been performed for different fractional orders by using MATLAB software. Two and three cases are discussed. The solution obtained dissertation are new helpful. In this study we added new solutions and graphs for dynamical behavior of Boussinesq fractional and KdV equation that are helpful in different physical phenomena exhibited by Boussinesq fractional and KdV equation.

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